Cryptography from Rings

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University of Michigan

HEAT Summer School
13 Oct 2015

Agenda

1. Polynomial rings, ideal lattices and Ring-LWE
2. Basic Ring-LWE encryption
3. Fully homomorphic encryption

Selected bibliography:

LPR'10 and '13 V. Lyubashevsky, C. Peikert, O. Regev.
"On Ideal Lattices and Learning with Errors Over Rings," Eurocrypt'10 and JACM'13.
"A Toolkit for Ring-LWE Cryptography," Eurocrypt'13.

BV'11 Z. Brakerski and V. Vaikuntanathan.
"Fully Homomorphic Encryption from Ring-LWE..." CRYPTO'11.

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2010 Ring-LWE: very efficient encryption, worst-case hardness ()

Cyclotomic Rings

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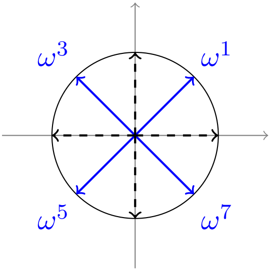

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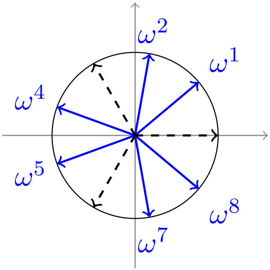

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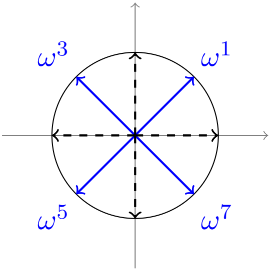

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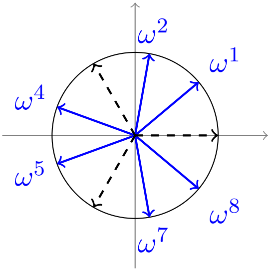

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- There are other $\mathbb{Z}$-bases, e.g., $\{\zeta_p^0, \ldots \zeta_p^{k-1}, \zeta_p^{k+1}, \ldots, \zeta_p^{p-1}\}$.

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- Bottom line: we can efficiently reduce operations in R to independent operations in prime-power cyclotomics $\mathbb{Z}[\zeta_{m_i}]$.

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This yields the "expansion" bound

$$\|a \cdot b\|_2 \leq \|a\|_\infty \cdot \|b\|_2 , \quad \text{where } \|a\|_\infty = \max_i |\sigma_i(a)|.$$

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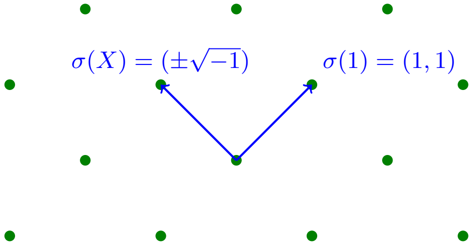

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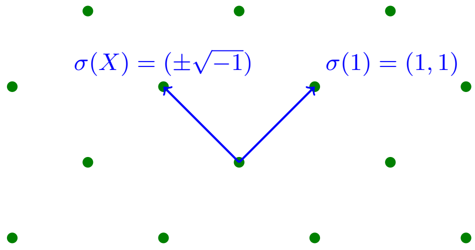

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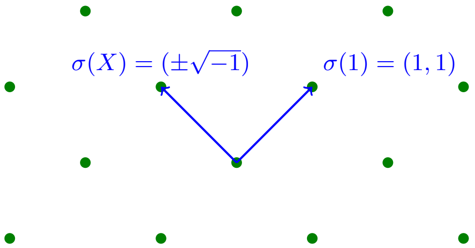

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- ✔ For any j, $\|X^j\|_2 = \sqrt{n}$ and $\|X^j\|_\infty = 1$.
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  So power & canonical geometries are equivalent (up to $\sqrt{n}$ scaling).

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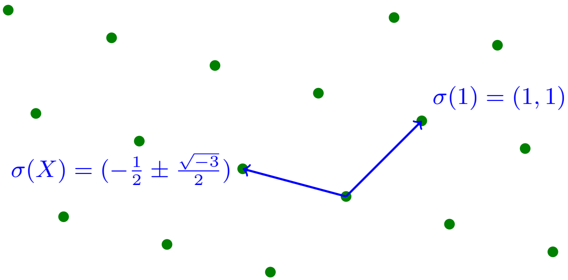

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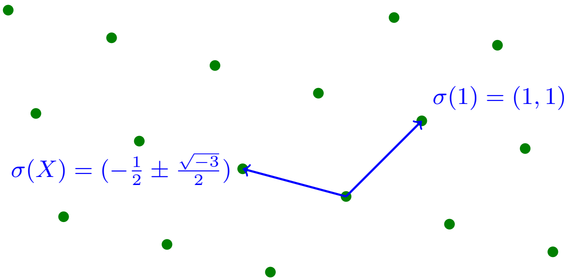

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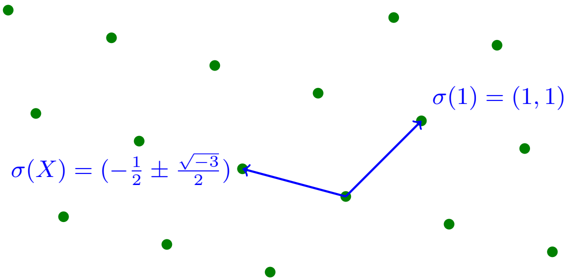

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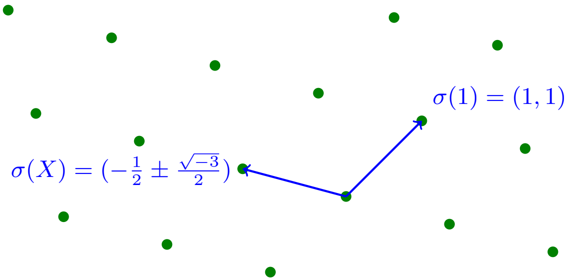

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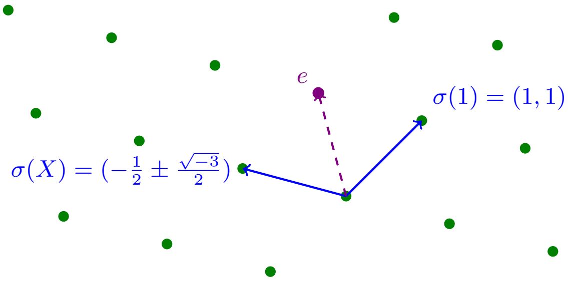

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  E.g., $e = 1 + X + \cdots + X^{p-2}$ but $\|e\| = \|1\| = \|X\| = \sqrt{p-1}$.

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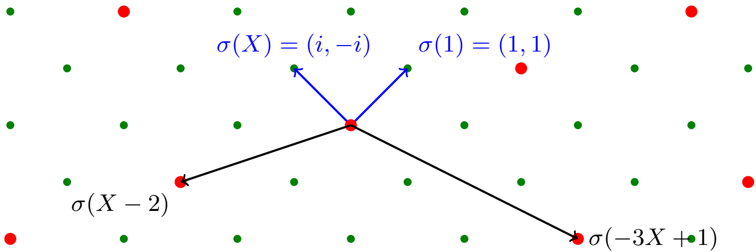

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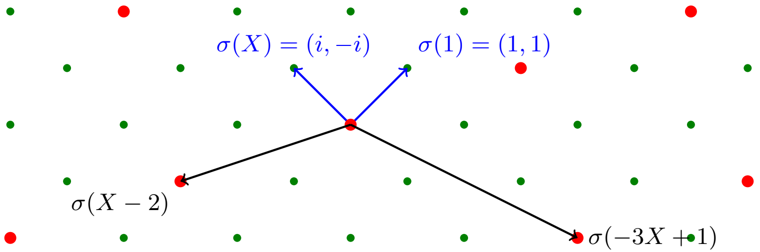

(Approximate) Ideal Shortest Vector Problem

- Given a $\mathbb{Z}$-basis of an ideal $\mathcal{I} \subseteq R$, find a nearly shortest nonzero $a \in \mathcal{I}$.

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  (For product $q = q_1 \cdots q_t$ of distinct primes, $R_q \cong R_{q_1} \times \cdots \times R_{q_t}$.)

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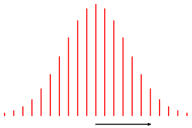

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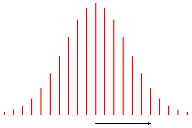

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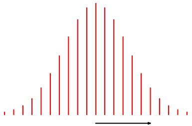

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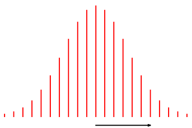

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- **Decision**: distinguish $(a_i\ ,\ b_i)$ from uniform $(a_i\ ,\ b_i) \in R_q \times R_q$.

Hardness of Ring-LWE [LyubashevskyPeikertRegev'10]

- Two main theorems (reductions):

worst-case approx-SVP on *ideal* lattices $\leq$ search Ring-LWE $\leq$ decision Ring-LWE

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  - ⋆ If you can break the crypto, then you can distinguish (a_i, b_i) from (a_i, b_i)...

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Alternative Interpretation

- Encryption of $\mu \in R_2$ is a linear polynomial $c(S) = c_0 + c_1 S \in R_q[S]$:
  1. $c(s) = e \approx 0 \bmod qR$, and
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Fully Homomorphic Encryption [BrakerskiVaikuntanathan'11]

- Need a system where: if c, c' encrypt m, m', then

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- Error size and polynomial degree (in S) grow with $\boxplus, \boxdot$.
  Use "linearization/key switching" and "modulus reduction" to shrink.